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Monterey, California

THESIS

THE FLAMING DATUM PROBLEM WITH VARYING SPEED

by

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June 2000

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13. ABSTRACT *(maximum 200 words)*

The problem of detecting an enemy submarine whose possible position was revealed by the hit of a torpedo is known as the "Flaming Datum" problem. All previous studies devoted to this theme make unrealistic assumptions about the speed of the escaping target when dealing with a diesel-electric submarine. In this kind of submarine the constraint imposed by the remaining charge of its batteries determines that its behavior is essentially conservative in how fast it should escape.

The objective of this thesis is to explore the idea of varying speed in the flaming datum problem. Two different approaches are considered. An analytical model is developed based on the relationship among some of the physical factors that could determine or constrain the behavior of a diesel submarine while escaping from the area of the flaming datum. The second approach considers a discrete event simulation using the Java-based Simkit package. Data analysis is used to determine a possible fit for the simulation results. Several tactics are explored to determine their effects on detection probability.

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THE FLAMING DATUM PROBLEM WITH VARYING SPEED

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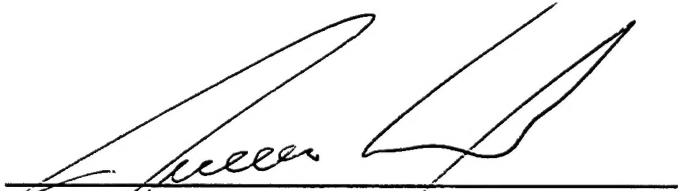Submitted in partial fulfillment of the
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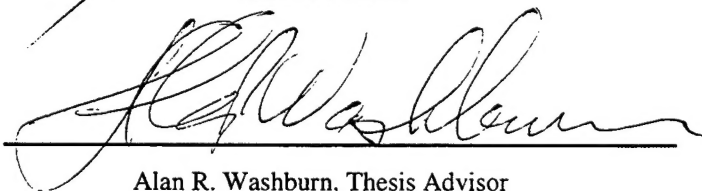
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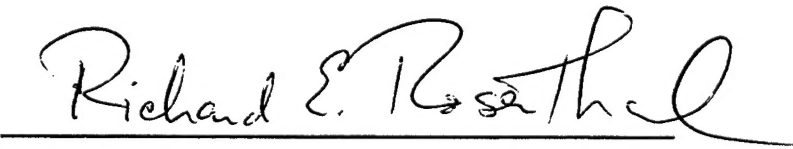
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ABSTRACT

The problem of detecting an enemy submarine whose possible position was revealed by the hit of a torpedo is known as the "Flaming Datum" problem. All previous studies devoted to this theme make unrealistic assumptions about the speed of the escaping target when dealing with a diesel-electric submarine. In this kind of submarine the constraint imposed by the remaining charge of its batteries determines that its behavior is essentially conservative in how fast it should escape.

The objective of this thesis is to explore the idea of varying speed in the flaming datum problem. Two different approaches are considered. An analytical model is developed based on the relationship among some of the physical factors that could determine or constrain the behavior of a diesel submarine while escaping from the area of the flaming datum. The second approach considers a discrete event simulation using the Java-based Simkit package. Data analysis is used to determine a possible fit for the simulation results. Several tactics are explored to determine their effects on detection probability.

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EXECUTUVE SUMMARY

The problem of detecting an enemy submarine whose possible position has been revealed by the hit of a torpedo on a surface vessel is known as the "Flaming Datum" problem. Since World War II, several studies have been devoted to the issue where a submarine tries to escape from the datum area while being hunted by a helicopter equipped with variable depth sonar (VDS) and/or antisubmarine weapons. All these studies make assumptions about the speed of the escaping target that are unrealistic when dealing with a diesel-electric submarine. The constraint imposed by the remaining charge of its battery determines that the behavior of this kind of submarine is essentially conservative in how fast it should escape. Even though the submarine is able to maintain a constant speed for long periods of time, this is only valid for very low speeds. Low speeds are not tactically acceptable for a submarine trying to evade a searcher, especially in the first minutes after the attack when operating in the vicinity of the Datum.

According to opinions and experiences collected by the author, a diesel-electric submarine will tend to escape, as a general rule, as soon as possible from the zone where an attack recently occurred. This tactic is motivated by the efforts of the enemy to find the submarine, but it is only sustainable for short periods of time depending on the characteristics of the battery, and its remaining charge.

The objective of this thesis is to develop and test a model for the Flaming Datum problem with a submarine trying to evade at varying speeds. First, the author develops an analytical model based on the existing relationship among some of the physical factors that determine or constrain the speed and behavior of a diesel submarine trying to escape from the area of the flaming datum. This model predicts the probability of detecting the

submarine as a function of the helicopter's delays in arriving at the datum area. To obtain a more accurate result for the same problem a discrete event simulation using the Java based Simkit package is developed and validated as believable. This simulation is also used to test some variations where either the submarine or the helicopter tries a specific tactic.

Even though, both the analytical model and the simulation show some weaknesses that may affect the credibility of the results, the possible sources of inaccuracy are identified and explained. It turns out that the submarine speed should be a strong function of time, rather than constant as usually supposed.

This study represents a more realistic method of posing the Flaming Datum problem when dealing with diesel-electric submarines that must respect an energy constraint. This constraint was never considered in any other study. For this reason the developed models may be considered a valid starting point for future works.

ACKNOWLEDGMENT

The author wants to thank Prof. Alan Washburn for his remarkable patience, time and guidance in performing this thesis. To the officers of the Chilean Navy Submarine Force for their support in my understanding of the meaning of sailing under the sea and specially to my wonderful wife Anita, for her unbounded comprehension, support and love.

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I. INTRODUCTION

A.   DESCRIPTION OF THE PROBLEM

The problem of detecting an enemy submarine whose possible position has been revealed by the hit of a torpedo on a surface vessel is known as the "Flaming Datum" problem. Since World War II, several studies have been devoted to the issue where a submarine tries to escape from the datum area while being hunted by a helicopter equipped with variable depth sonar (VDS) and/or antisubmarine weapons. All these studies make assumptions about the speed of the escaping target that are unrealistic when dealing with a diesel-electric submarine. The constraint imposed by the remaining charge of its battery determines that the behavior of this kind of submarine is essentially conservative in how fast it should escape. Even though the submarine is able to maintain a constant speed for long periods of time, this is only valid for very low speeds. Low speeds are not tactically acceptable for a submarine trying to evade a searcher, especially in the first minutes after the attack, when operating in the vicinity of the Datum.

According to opinions and experiences collected by the author, a diesel-electric submarine will tend to escape, as a general rule, as soon as possible from the zone where an attack recently occurred. This tactic is motivated by the efforts of the enemy to find the submarine, but it is only sustainable for short periods of time depending on the characteristics of the battery, and its remaining charge.

The objective of this thesis is to develop and test a model for the Flaming Datum Problem with a submarine trying to evade at varying speeds. To do so, the author develops an analytical model based on the existing relationship among some of the

1

physical factors that determine or constrain the speed and behavior of a diesel submarine trying to escape from the area of the flaming datum. The model is tested with a discrete event simulation.

B. BACKGROUND

The previous studies can be divided into those that have been used to solve the problem as either one-dimensional or two-dimensional games. In the first group, Meinardi [Ref. 1] considers a multi-staged search game where the submarine hides in a row of n_o boxes. At each stage, the helicopter chooses a box and inspects it. If the target is not found, it can move on to an adjacent box or remain in the same one. Both target and searcher keep track of the searched boxes. The time spent to transit among the boxes is not taken into consideration, and thus, the applications of this game are limited for further analysis.

Baston and Bostock [Ref. 2] develop a game that could be considered a continuous version of the flaming datum problem in one-dimensional space. The submarine moves in a long narrow channel and the helicopter searches for it by carrying a limited number of bombs. Each bomb has a lethal radius R and a limited number of bombs are available. These two characteristics can be considered analogous to the sonar detection range and the number of dips available. In this game the submarine and the helicopter are restricted in movement by their maximum speeds (U and V) and instantaneous changes in velocity are allowed.

The second group deals with two-dimensional cases. Danskin [Ref. 3] supposes that the datum is known precisely and the submarine strategy is to choose a course θ and

a speed v with $0 <= v <= v_o$, where v_o is the maximum speed. It holds to both throughout the search and this establishes the submarine's position within its speed circle [Figure 1]. Danskin also supposes that the submarine does not know what the helicopter is doing, so that it does not need to alter its speed or course. The helicopter dips are assumed to be inside the submarine's speed circle and non-overlapping. Danskin's assumptions seem to be very drastic.

Cheong [Ref. 4] develops a game where the submarine and helicopter are assumed to be moving at the constant speeds of U and V respectively. However, the submarine is allowed to execute evasive maneuvers by moving directly away from the last helicopter dip as well as moving away from the datum. Meanwhile the helicopter conducts a random search.
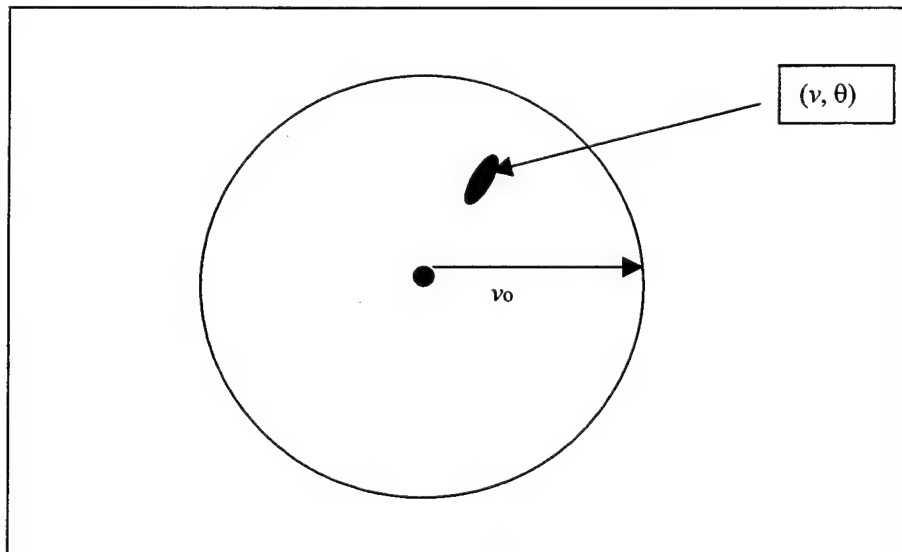
Figure 1. Danskin approach. Submarine speed circle, showing a choice of strategy. (From Ref. 3, p. 511)

Washburn [Ref. 5, p 2-7] states that expected number of detections for a random search in an expanding disk can be modeled as:

$$\int_{\tau}^{\infty} \frac{2RV}{\pi y(t)^2} dt \qquad \text{(Equation 1)}$$

where V is the searcher's speed, R is the range of its cookie cutter sensor, τ is the delay of the helicopter arriving at the datum, and $y(t)$ is the expanding disk radius at time t, which is dependent on the submarine's maximum speed. Equation 1 will be used in developing an analytical model in Chapter II.

Washburn and Thomas [Ref. 6] model the flaming datum problem as a dynamic search game. They relax the rules of motion for the submarine, so that its future position could be anywhere inside its speed circle and not necessarily connected to the previous position. This assumption, in a sense, permits an unbounded target speed. In fact, the maximum speed is only considered to generate the radius of the area to be searched for, or the maximum speed circle. Even though this assumption appears to favor the submarine, Washburn and Thomas are of the opinion that it is of minor consequence. The authors compare the similarity of their results with Cheong's, and state that this similarity might imply that both models are actually wrong. According to Washburn "no Flaming Datum Problem has ever been solved or even usefully bounded, so the required standard of comparison is missing" [Ref 6. P 421].

C.   APPROACH

Our first approach is to use an analytical model to develop a plausible speed profile for the submarine. This model is based on the existing relationship among some of the physical factors that determine or constrain the speed and behavior of a diesel submarine trying to escape from the area of the flaming datum. Parameters and variables such as power, energy, speed, speed sustainability and time are employed to determine an expression to compute the probability of detection of the submarine as a function of time. The submarine's foe is assumed to be a generic helicopter equipped with variable depth sonar (VDS) with a "cookie-cutter" sensor. This helicopter arrives at the flaming datum after some delay τ and has a limited endurance. If the submarine is not detected after this period, it will be considered to have escaped.

In a second and more accurate approach a Monte Carlo discrete event simulation using the Java-based Simkit Software package is developed. A submarine with a set of continuous decreasing velocity functions that are consistent with the capabilities of a generic submarine used in the analytical model is considered.

A detailed description of the assumptions and strategies for the submarine and helicopter are given in the next chapters.

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II. ANALYTICAL MODEL

A.   ASSUMPTIONS

As mentioned in Chapter I, all previous studies make assumptions about the speed of the target that are unrealistic when dealing with a diesel-electric submarine. The constraint imposed on this kind of submarine by the remaining charge of the batteries forces its behavior to be essentially conservative when considering escape speed. We make the following assumptions:

1.   Submarine Assumptions

    a. The submarine will normally try to escape as quickly as possible in the first few minutes in order to make the FOC (farthest on circle) larger, and to increase the probability of escape. This behavior is justified since the first reaction of an attacked force will be to verify that the submarine is no longer in the vicinity of the datum.

    b. The battery is initially close to having a maximum charge. This is a reasonable assumption for a deliberate attack.

    c. The submarine is able to estimate the expected helicopter's delay in arriving in the datum area.

    d. The depth factor will be ignored.

2. Helicopter Assumptions

a. The position of the datum is known with no error.

b. The searcher knows the submarine's capacities related to its speed. This assumption is considered valid when taking into account that open sources and commercial manuals are readily available. As a general rule, the searcher should know who the possible enemies are.

c. The helicopter moves at a constant speed when not dipping. The times to dip and raise the sonar, and search in a position, are also constant.

d. The helicopter does not assume any particular behavior for the submarine; therefore its dips are uniformly distributed in the submarine's maximum possible distance circle (MPD).

B.   THEORETICAL FOUNDATION

Rydill and Burcher [Ref 7, p 291], state that the effective power (P_E) needed to "tow" the submarine at speed $u(t)$ in a submerged condition is given by:

$$P_E = \frac{1}{2} \rho u(t)^3 S_{REF} C_{TS} \qquad (Equation\ 2)$$

where S_{REF} is the wetted area of the bare hull and casing plus the wetted area of all appendages minus the area covered by the appendages, C_{TS} is the total resistance coefficient or drag coefficient and ρ is the sea water density.

Now let

$$J = \frac{1}{2} \rho S_{REF} C_{TS}$$

Thus, the energy of the submarine's batteries required to move a submarine with a speed $u(t)$ during an interval $[0, t]$ can be expressed as:

$$E(t) = J \int_0^t u(x)^3 dx \qquad (Equation\ 3)$$

Then, the constraint imposed by the energy of the batteries if the submarine wants to move for a given period $[0, T]$ with a speed $u(t)$ is given by :

$$E(T) \leq E_{TOT} \qquad (Equation\ 4)$$

E_{TOT} is the total battery energy when the submarine begins to escape at time 0. This energy, according to assumption b), should be very close to maximum charge. E_{TOT} would be normally measured in [KWHR], but for convenience we choose units for energy

such that $J = 1$ in Equation 3. Therefore, we can compute E_{TOT} as U^3 multiplied by the endurance at this speed, where U is any arbitrarily chosen speed. A realistic value for E_{TOT} for a modern diesel-electric submarine is:

$$(16.4 \ mph)^3 \cdot (1.2 \ hr) = 5293 \ miles^3 hrs^{-2} \qquad (Equation \ 5)$$

C.   MATHEMATICAL FORMULATION

In Equation 1, the detection rate for a random search in an expanding disk is modeled as $(VW)/(\pi y(t)^2)$. Therefore the average number of detections in the time interval $[\tau, T]$ when the search is continuous is:

$$\int_\tau^T \frac{\alpha}{y(t)^2} dt$$

where $\alpha = 2RV/\pi$.

However our helicopter searches by dipping in an expanding circle, so the value of alpha will depend on the position of the dips and will be changing as a function of time. Therefore a better expression for $\alpha(t)$ is:

$$\alpha(t) = \frac{rate \ of \ area \ \mathrm{covered}}{\pi} = \frac{\pi R^2}{\pi \left(\delta + \dfrac{Cy(t)}{V} \right)} = \frac{R^2}{\left(\delta + \dfrac{Cy(t)}{V} \right)} \qquad (Equation \ 6)$$

where C is a factor near 1 that allows computing the average distance to be flown from dip to dip and δ is the time to make a dip. If we make $C=1$, the average number of detections in the time interval $[\tau, T]$ becomes:

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$$\int_\tau^T \frac{R^2}{\left(\delta + \dfrac{y(t)}{V}\right)y(t)^2} dt = \frac{R^2}{\delta} \int_\tau^T \frac{dt}{y(t)^2 + \dfrac{y(t)^3}{\delta V}} = K \int_\tau^T \frac{dt}{y(t)^2 + \dfrac{y(t)^3}{\delta V}}$$

where $K = \dfrac{R^2}{\delta}$

A logical objective function for the submarine is to minimize the expected number of detections during the interval $[\tau, T]$.

PROGRAM 1

Minimize

$$\int_\tau^T \frac{\alpha(t)}{y(t)^2} dt$$

Subject to

$$\int_0^T u(t)^3 dt \le E_{TOT} \text{ and } y(0) = 0$$

Before time τ the submarine should escape with constant speed $y(\tau)/\tau$. Therefore the energy constraint for Program 1 is:

$$\tau^{-2} y(\tau)^3 + \int_\tau^T u(t)^3 dt \le E_{TOT}$$

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This program can be expressed in terms of a Lagrangian unconstrained function as:

$$F\big(y(t),u(t),t\big)=\frac{\alpha(t)}{y(t)^2}+\lambda u(t)^3 \qquad\qquad (\textit{Equation } 7)$$

which is equal to

$$F\big(y(t),u(t),t\big)=\frac{K}{y(t)^2+\dfrac{y(t)^3}{\delta V}}+\lambda u(t)^3 \qquad\qquad (\textit{Equation } 8)$$

Euler's equation from the calculus of variations [Ref. 8, p 399], states that a necessary condition for optimality is:

$$F\big(y(t),u(t),t\big)-u(t)\frac{\partial F\big(y(t),u(t),t\big)}{\partial u(t)}=L \qquad\qquad (\textit{Equation } 9)$$

Euler also requires that

$$\frac{\partial F}{\partial u(T)}\big(y(T),u(T),T\big)=0\Rightarrow u(T)=0 \qquad\qquad (\textit{Equation } 10)$$

Then

$$L=\frac{K}{y(t)^2+\dfrac{y(t)^3}{\delta V}}+\lambda u(t)^3-3\lambda u(t)^3=\frac{K}{y(t)^2+\dfrac{y(t)^3}{\delta V}}-2\lambda u(t)^3;\ \ \tau\le t\le T \qquad (\textit{Equation } 11)$$

and solving for u(t)

$$u(t)=\left[\frac{1}{2\lambda}\left(\frac{K}{y(t)^2+\dfrac{y(t)^3}{\delta V}}-L\right)\right]^{1/3}\ ;\ \ \tau\le t\le T \qquad\qquad (\textit{Equation } 12)$$

If we make $A=\dfrac{K}{2\lambda}$ and $B=\dfrac{L}{2\lambda}$

we have

$$u(t) = B^{1/3} \left[\frac{\dfrac{A}{B}}{y(t)^2 + \dfrac{y(t)^3}{\delta V}} - 1 \right]^{1/3} \quad ; \ \tau \leq t \leq T \qquad (Equation\ 13)$$

If $\dfrac{A}{B} = Y^2$ and $B^{1/3} = v$, $u(t)$ becomes:

$$u(t) = v \left[\frac{Y^2}{y(t)^2 + \dfrac{y(t)^3}{\delta V}} - 1 \right]^{1/3} \quad ; \ \tau \leq t \leq T \qquad (Equation\ 14)$$

Y is an adopted constant with units of distance and v has units of speed.

Given $y(\tau)$, Equation 14 is a first order ordinary differential equation that can be solved

for y(t); $\tau \leq t \leq T$.

Equation 4 states that $E(T) \leq E_{TOT}$. For this reason the submarine's speed might

be any value less than or equal to $u(t)$. Therefore, the submarine's possible locations

could be considered uniformly distributed over the circle of radius $y(t)$ (FOC).

According to Washburn [Ref. 5, p. 2-10], the probability of detection as a function of

time for a random search, given $y(t)$, can be expressed as:

$$PD(t) = 1 - e^{-\int_\tau^t \frac{\alpha(x)dx}{y(x)^2}} = P(X \leq t); \tau \leq t \leq T \qquad (Equation\ 15)$$

where the random variable X is the time to detection.

Even though Equation 14 seems not to have an analytical solution, it is possible to

solve it numerically using the solver feature of the Excel spreadsheet. This is done by

minimizing the value of the integral $\int^{T} (\alpha(x)/y(x)^2)dx$ in Equation 15 at time T, subject

to the constraint imposed by Equation 4, and changing the values of $y(\tau)$, v and Y.

It is unknown whether this problem possesses the convexity properties that would

guarantee global optimality. However, when the problem was solved repeatedly using a

wide range of different starting values for the variables $y(\tau)$, v and Y, it always converged

to the same solutions. We suspect, but can not guarantee, this is a global optimum.

Equation 14 might be seen as a possible speed function for the submarine and

Equation 15 is the probability of detecting it when the search is executed inside the FOC.

However, the helicopter doesn't know what is the enemy is really doing, so it has to

search covering every possible target's behavior. The helicopter must assume that a

possible objective function for the submarine is to maximize $y(t)$ at every time t by

using all of E_{TOT}. This assumption is expressed in Program 2, which has t as parameter for

$0 \le t \le T$:

PROGRAM 2

Maximize

$y(t)$

Subject to

$\int_{0}^{t} u(x)^3 dx \le E_{TOT}$

Since the integrand is a convex function of u(x), the speed should be constant at $u^*(t)$ for

$0 \le x \le t$, and

$tu^*(t)^3 = E_{TOT}; \ 0 \le t \le T$ *(Equation 16)*

therefore

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$$u^*(t) = (E_{TOT}/t)^{1/3}; \ 0 \leq t \leq T \qquad \text{(Equation 17)}$$

Multiplying by t

$$u^*(t)t = y^*(t) = E_{TOT}^{1/3}t^{2/3}; 0 \leq t \leq T \qquad \text{(Equation 18)}$$

$y^*(t)$ is the radius of the MPD at time t.

For any given moment t, $0 \leq t \leq T$, $u^*(t)$ is the optimal constant speed before t to maximize the distance from the datum. The helicopter does not guess any particular value for t and makes the pessimistic assumption that for every moment in the interval $[0,T]$ the submarine is able to reach $y^*(t)$.

Now let $\dfrac{dy^*(t)}{dt} = U^*(t)$

Then,

$$U^*(t) = \frac{2}{3}E_{TOT}^{1/3}t^{-1/3} \qquad \text{(Equation 19)}$$

$U^*(t)$ is an infeasible "Virtual Speed" function to cover every possible use of the submarine's battery.

Finally, the probability of detection when the helicopter searches within the MPD can be expressed as

$$PD^*(t) = 1 - e^{-\int_\tau^t \frac{\alpha(x)dx}{y^*(x)^2}} =; \tau \leq t \leq T \qquad \text{(Equation 20)}$$

The following graphs show the evolution of some of the variables in Programs 1 and 2 when considering the following realistic data:

Constants:

- Helicopter's delay $\tau = 10$ minutes.

- Sweep width W = 3 NM.

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- Submarine energy $E_{TOT} = 5293 \ miles^3 hours^{-2}$.

- Helicopter's speed $V = 60$ knots.

- $\delta = 0.05$ hours.

- $T = 2.83$ hours.

Optimized Variables:

- Initial distance submarine-datum $y(\tau) = 4.2385$ NM.

- $Y = 65.7236$ NM.

- $v = 5.64847$ knots.

Figure 2 shows the distance $y(t)$ from the flaming datum to the FOC that minimizes the probability of detection for the submarine (Program 1), and the MPD considered in the helicopter's search (Program 2). Euler's method is used to solve Equation 14 numerically. The distance $y(t)$ is utilized to compute the submarine speed $u(t)$. Then, the computed speed is used to obtain the distance $y(t + \Delta t)$ considering that:

$y(t + \Delta t) = y(t) + u(t)\Delta t$, where $\Delta t = 0.02$ hours. (*Equation* 21)
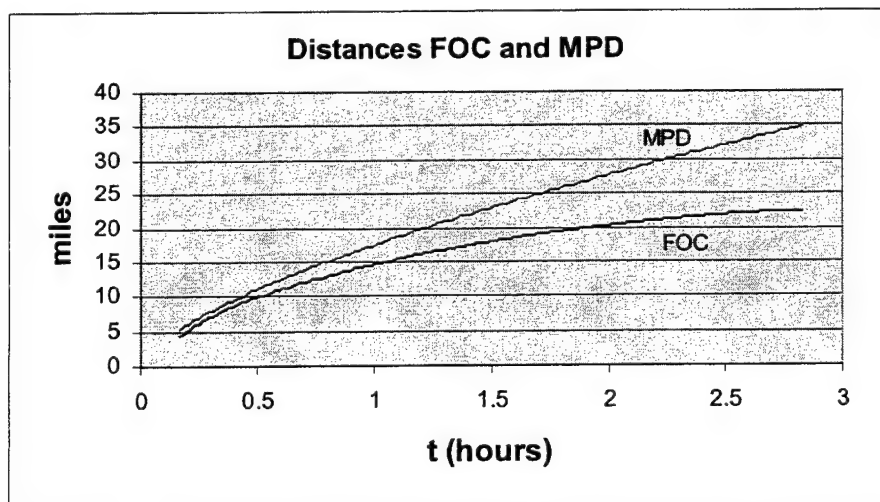
Figure 2. Submarine distance as a function of time.

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Figure 3 shows the submarine's velocity functions $U^*(t)$, $u(t)$ and $u^*(t)$. $u(t)$ is the velocity function to reach the FOC shown in Figure 2 (Equation 14). $U^*(t)$ is the submarine's virtual speed that determines the MPD assumed by the helicopter (Equation 19). Finally, for the function $u^*(t)$ every value is independently feasible, and each one implies the total consumption of the battery at time t (Equation 17).
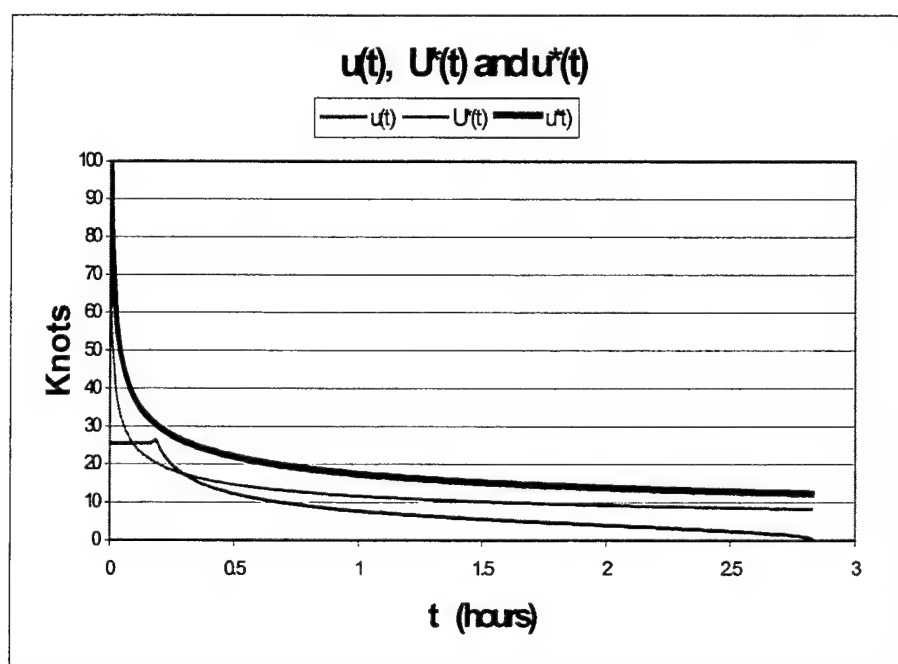
Figure 3. $U^*(t)$, u (t) and $u^*(t)$ when the helicopter's delay is 10 minutes and T= 2.82 hrs.

Figure 4 shows the cumulative distribution function (CDF) for the time to detection as a function of time when the helicopter's delay is 10 minutes and the search is executed:

- Inside the MPD circle (Equation 20).

- Inside the FOC (Equation 15).
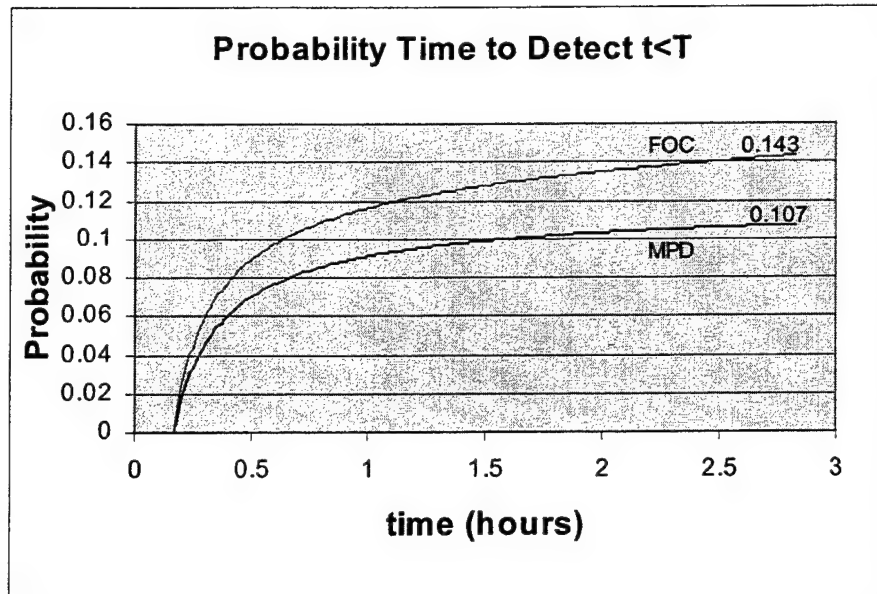
Figure 4. Detection probability if helicopter's delay is 10 minutes.

Figure 5 shows the decrease in probability of detection as a function of the helicopter's delay τ, if the search is executed inside either the FOC or the MPD. This graph is obtained from the analytical model for the parameters listed in page 15.
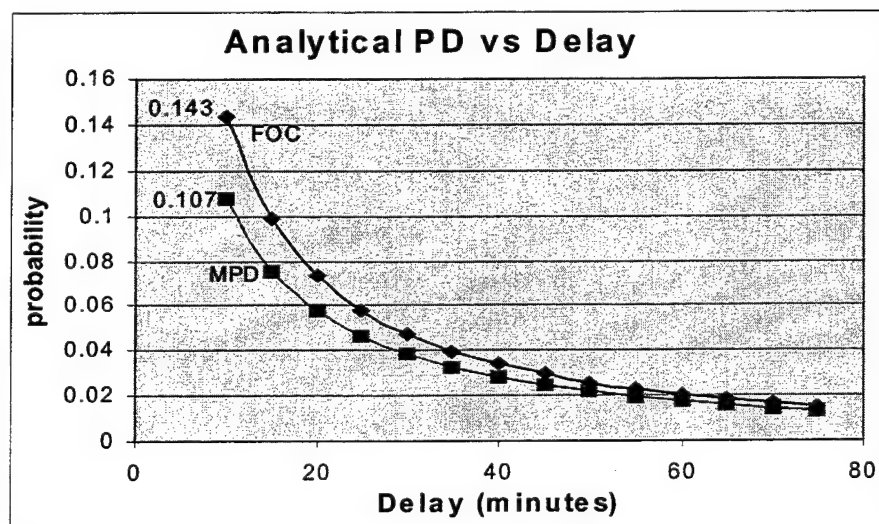
Figure 5. Analytical Probability of Detection versus Helo's delay.

To obtain a more accurate result a discrete event simulation using the Java-based Simkit Software package is developed next, which considers the same characteristics for the submarine and helicopter used in the analytical model. The next chapter shows the methodology used to solve some of the problems related to the movement and strategies for the searcher and the target.

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III. THE SIMULATION

A. CONSIDERATIONS

1. Submarine Strategy

a) *Submarine Initial Position*

The initial distance from the datum $y(\tau)$ is an optimized variable for each value of τ. To determine the initial position of the submarine for each replication, the distance $y(\tau)$ is multiplied by a different right triangular random variate $[0, 1]$ called ϖ. The pair $(y(\tau)\varpi, \theta)$ is then transformed to rectangular coordinates and thus creates an initial position uniformly distributed inside the circle of radius $y(\tau)$.

b) *Velocity function*

The submarine's general velocity function is generated through Equation 14

$$u(t) = v \left[\frac{Y^2}{y(t)^2 + \frac{y(t)^3}{\delta V}} - 1 \right]^{1/3} \quad ; \quad \tau \leq t \leq T$$

where Y and v are optimized constants for the particular helicopter's delay τ and $y(t)$ is the submarine's distance from the datum at time t, obtained from Equations 14 and 21. A vector holds the velocity whose value is updated every 9 seconds in order to create a function which is relatively continuous. For each replication, the speed function is multiplied by the same ϖ used to determine the submarine's initial position. In doing so, a different speed behavior is generated for each simulation run and it is consistent with assumption a) for the submarine in Chapter II.

c) Course

The submarine flees from the datum with a random course θ, restricted turn capability and executing 9-second legs. When a leg has finished, the submarine decides if it will change course. This decision is made by sampling a uniform [0,1] random variate and comparing it with a reference value. This reference value is very close to 1 and the maximum change in course is restricted to 3 degrees port or starboard. In doing so, the submarine maintains a fleeing attitude.

2. Helicopter Strategy

a) Position of the First Dip

The helicopter's first dip is uniformly distributed in the circular area whose radius is $y^*(\tau + time\ to\ dip)$, where $y^*(t)$ is the MPD function obtained from Equation 18.

b) Position of the Next Dip

The helicopter dips uniformly in the area whose radius at time t' is $y^*(t')$. Washburn and Thomas [Ref. 6,p. 420] state that, considering X as the present helicopter's position in the unit circle and t as the present time, the earliest time t' to reach a point X' in the same circle guaranteeing uniform search can be computed as:

$$t' = t(d - abc + e)/(1 - b^2) \equiv T(X, X', t) \qquad (Equation\ 22)$$

Where

$$a = \frac{U\,|\,X\,|}{V} \quad b = \frac{U\,|\,X'|}{V} \qquad c = Cos(angle\ between\ X\ and\ X')$$

$$d = 1 + \frac{D}{t} \qquad and \qquad e^2 = a^2 + b^2 d^2 - a^2 b^2 (1 - c^2) - 2abcd$$

U and V are the submarine and helicopter's velocity and D is the time required for the helicopter to make a dip with the sonar.

The result stated in Equation 22 makes it possible to compute the position of the helicopter's next dip and to be certain that it will be uniformly distributed in the submarine's possible positions disk. Even though this procedure is applicable when dealing with a submarine with a constant maximum speed, a similar approach can be done when considering that for any time t, the submarine's virtual speed value $U^*(t)$ may be obtained from Equation 19.

Let
$$X = [x, y] \quad X' = [x', y']$$
be the helicopter's position coordinates in the unit circle at time t and t' respectively, so $x^2 + y^2 \leq 1$ and $x'^2 + y'^2 \leq 1$.

Then, the time for the next dip t' should satisfy

$$t + D + \frac{\sqrt{[x'y^*(t') - xy^*(t)]^2 + [y'y^*(t') - yy^*(t)]^2}}{V} = t'; \quad (Equation\ 23)$$

Where $\sqrt{[x'y^*(t') - xy^*(t)]^2 + [y'y^*(t') - yy^*(t)]^2}$ is the distance between the helicopter's present dip position at time t and the future dip position at time t'. Even though Equation 18 allows obtaining the value of $y^*(t')$, replacing it in the equation above creates a sixth degree equation that is not useful to solve for t'. However, we can obtain the approximate distance between [x, y] and [x', y'] if we assume that an approximation for $y^*(t')$ is:

$$y^*(t') = y^*(t) + U^*(t)(t'-t)$$

where $U^*(t)$ is the virtual speed introduced in Equation 19.

Replacing $y^*(t')$ in Equation 23 we have:

$$t + D + \frac{\sqrt{[x'(y^*(t) + U^*(t)(t'-t)) - xy^*(t)]^2 + [y'(y^*(t) + U^*(t)(t'-t)) - yy^*(t)]^2}}{V} = t'$$

If both sides of the equation are multiplied by V, then

$$Vt + VD + \sqrt{[x'(y^*(t) + U^*(t)(t'-t)) - xy^*(t)]^2 + [y'(y^*(t) + U^*(t)(t'-t)) - yy^*(t)]^2} = Vt'$$

and

$$V(t + D - t') = -\sqrt{[x'(y^*(t) + U^*(t)(t'-t)) - xy^*(t)]^2 + [y'(y^*(t) + U^*(t)(t'-t)) - yy^*(t)]^2}$$

Squaring and reducing, then

$$V^2(t + D - t')^2 = [x'(y^*(t) + U^*(t)(t'-t)) - xy^*(t)]^2 + [y'(y^*(t) + U^*(t)(t'-t)) - yy^*(t)]^2$$

Now let

$$t + D = a, \quad y^*(t)(x'-x) - x'U^*(t)t = b, \quad \text{and} \quad y^*(t)(y'-y) - y'U^*(t)t = c.$$

Then

$$V^2(a - t')^2 = [b + x'U^*(t)t']^2 + [c + y'U^*(t)t']^2$$

and reducing

$$V^2a^2 - 2aV^2t' + V^2t'^2 = b^2 + 2bx'U^*(t)t' + (x'U^*(t))^2t'^2 + c^2 + 2cy'U^*(t)t' + (y'U^*(t))^2t'^2$$

Now factorizing

$$t'^2\left(V^2 - (x'U^*(t))^2 - (y'U^*(t))^2\right) - t'\left(2aV^2 + 2bx'U^*(t) + 2cy'U^*(t)\right) + V^2a^2 - b^2 - c^2 = 0$$

(Equation 30)

If

$$d = V^2 - (x'U^*(t))^2 - (y'U^*(t))^2, \qquad e = 2aV^2 + 2bx'U^*(t) + 2cy'U^*(t)$$

and $f = V^2a^2 - b^2 - c^2$

then, $dt'^2 - et' + f = 0$ *(Equation 31)*

which is a normal second degree equation. Finally

$$t' = \frac{e \pm \sqrt{e^2 - 4df}}{2d}$$ *(Equation 32)*

In practice we choose the smallest positive root of Equation 32, if greater than the current time t. The chosen time will be the solution for the earliest time that a point in the unit circle can be reached to guarantee a uniform dip location.

Once t' is obtained, $y^*(t')$ can be calculated using Equation 18. Therefore, the coordinates of the next helicopter's dip are:

$$X' = \left(x'y^*(t'),\ y'y^*(t')\right)$$

Figure 6 shows two typical sequences of dip times when the helicopter's delay is 10 minutes.
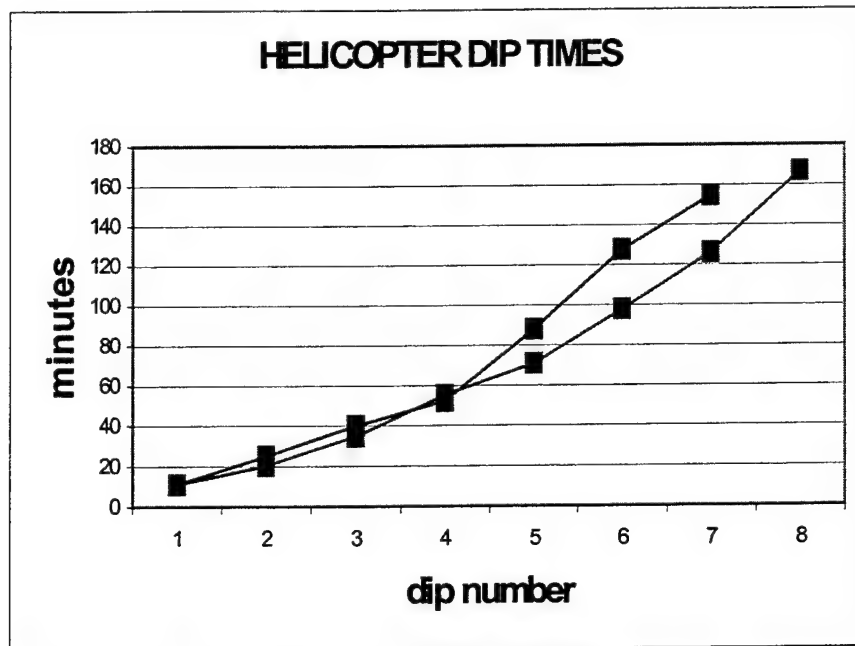
Figure 6. Helicopter's typical dip times when delay is 10 minutes.

3. Simulation Inputs

a) Parameters

In Equation 14, the parameters Y, v, and $y(\tau)$ are optimized for different

values of τ using a spreadsheet and a non-linear programming solver to solve Program 1,

as explained in Chapter II C. The following table shows the parameters introduced in the

simulation for each value of τ. All the columns except for the first were calculated based

on the following constants:

- Helicopter dip radius $R = 1.5$ NM.

- Submarine energy $E_{TOT} = 5293\ miles^3 hours^{-2}$.

- Helicopter speed $V = 60$ knots.

- Time to make a dip $\delta = 0.05$ hours.

- Δt for the spreadsheet $= 0.02$ hours.

-   Helicopter endurance = 2.65 hours.

PARAMETERS INTRODUCED IN SIMULATION			
Delay	Initial Distance	v	Y
10	4.238500882	5.64847041	65.72360358
15	5.677708987	6.00920271	73.40442537
20	6.981682236	6.2818038	79.96614511
25	8.1932953	6.50380744	85.87639327
30	9.340396291	6.72857452	90.75760417
35	10.43320542	6.87398188	96.13915272
40	11.48289288	7.01625994	101.0523998
45	12.49630944	7.14253496	105.7943132
50	13.47854091	7.2559127	110.3977112
55	14.43347615	7.3589631	114.88283
60	15.36430179	7.45277939	119.2756334
65	16.2685256	7.51082216	124.0679351
70	17.16295785	7.61865811	127.8289081
75	18.03435462	7.69262138	132.0131104

Table 1. Simulation Parameters.

4.   Simulation Validation

a)   Average PD (PD$_A$)

The average probability of detection (*PD*) for a particular value of the

helicopter's delay τ is obtained if it is considered that for a helicopter dip at time t, the

probability of detection might be computed as:

$$PD(i^{th} \ dip) = \frac{R^2}{y^*(t)^2} \qquad\qquad (Equation\ 33)$$

Let N be the number of dips made by the helicopter, including the detecting dip (if any)

but none beyond it. Then the probability of detection *PD* for any simulation replication

is:

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$$PD = E\left(\sum_{i=1}^{N} PD(i^{th} \ dip)\right) \qquad \textit{(Equation 34)}$$

The expected value is estimated by replication. Letting N_k be the number of dips in the

k^{th} replication and t_{ki} be the time of the ith dip in the kth replication, the estimate for Q

replications is:

$$\textit{Average } PD_A(\tau,Q) = \frac{\sum_{k=1}^{Q}\sum_{i=1}^{N_k} \dfrac{R^2}{y^*(t_{ki})}}{Q} \qquad \textit{(Equation 35)}$$

B.   BASIC RESULTS

10000 replications were run for 14 different values of τ. The simulation

simultaneously computed Monte Carlo estimates for the value of PD_A. Figure 7 (next

page) shows the evolution of the probability of detection and its 95 % confidence

intervals versus τ as estimated by the simulation compared with PD_A, when the

helicopter's search is executed inside the MPD. It can be seen that the values of the

estimates PD_A remain inside the confidence intervals for all values of τ, confirming the

accuracy of the simulation results. In the next chapter the simulation is used to predict the

probability of detection when the behavior of the submarine and the helicopter are varied,

for helicopter's delay of 10, 20 and 30 minutes.

Tables with detailed simulation results used in this chapter are included in

Appendix A.
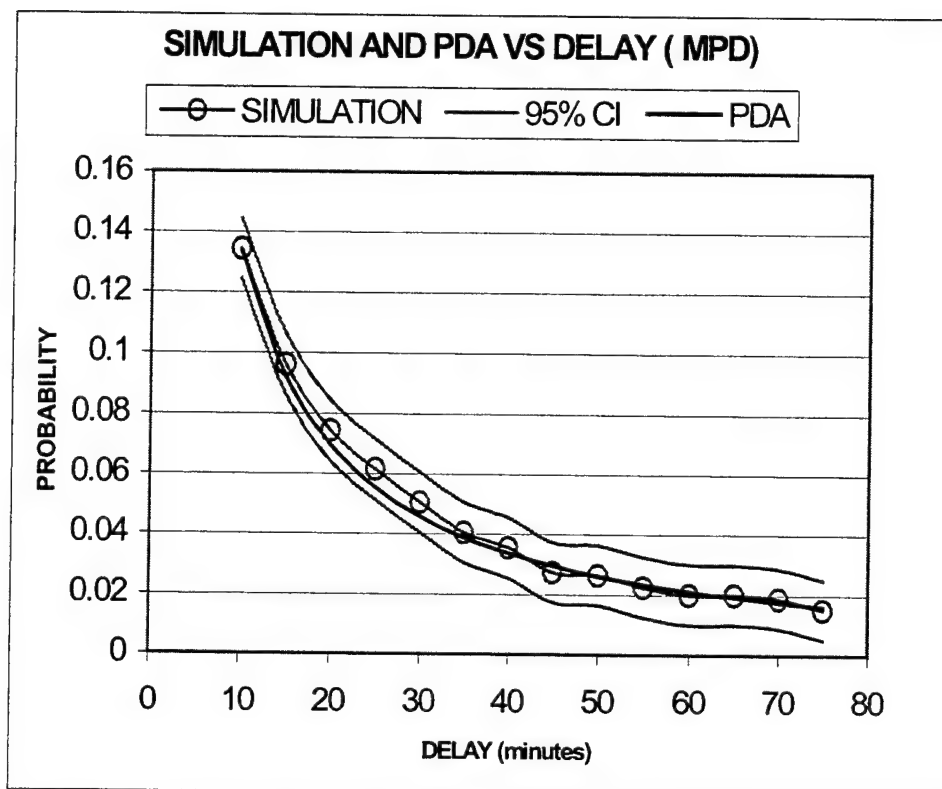
Figure 7. Probability of Detection obtained from simulation and estimate *PD_A* versus helicopter's delay, when search is inside MPD.

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IV. RESULTS AND ANALYSIS

A. MODEL FITTING

Figure 8 shows the probability of detection as a function of the helicopter's delay τ according to the results obtained from the simulation, when the search is executed inside the MPD. A fitted function is also shown and the corresponding regression analysis is detailed in Appendix B. That curve is valid for the parameters listed in chapter III.

SIMULATION AND FITTED FUNCTION (MPD)

—○— SIMULATION

$y = -9\text{E-}07x^3 + 0.0002x^2 - 0.009x + 0.2044$

$R^2 = 0.9935$

Figure 8. Simulation results and fitted function.

B.   TACTICAL IMPLICATIONS

Different variations to the problems were explored. The following results were obtained from 10000 simulation replications. A large-sample $100(1-\alpha)\%$ confidence interval for a population proportion p, if n is the sample size is:

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

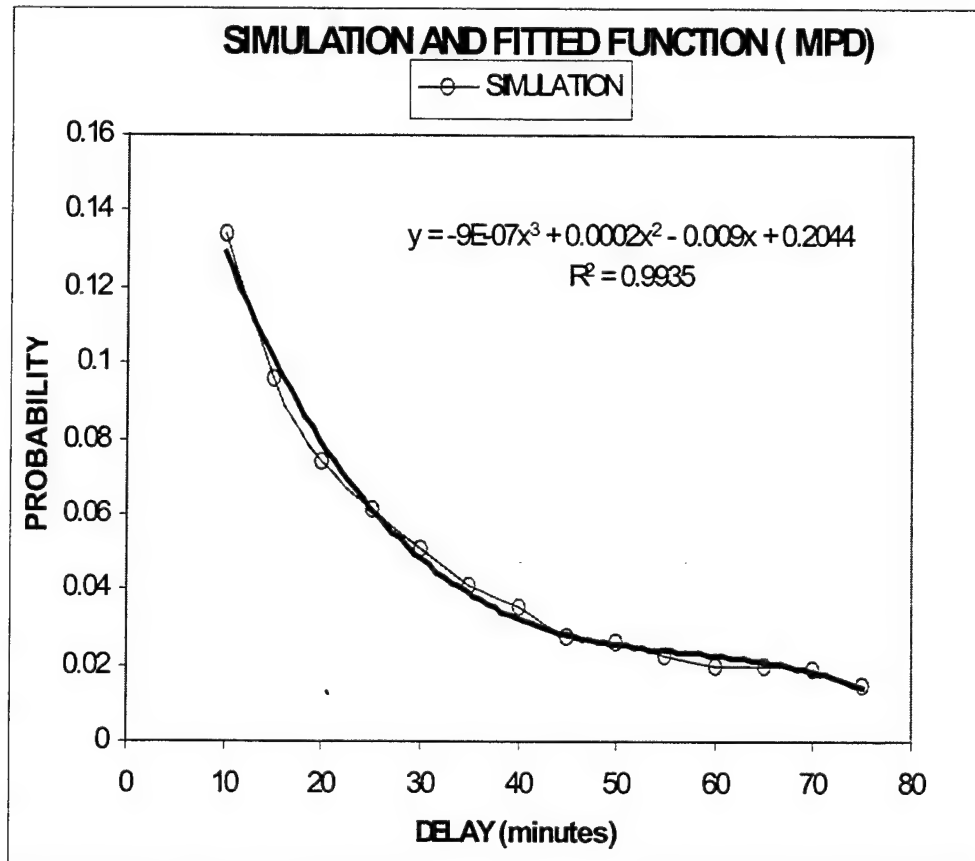When we don't know the value of $\hat{p}$ the most conservative approach is to use $\hat{p} = .5$, to maximize the value of the expression under the root [Ref. 9, p.289]. Therefore a 95% confidence interval is:

$$\pm 1.96\sqrt{\frac{0.5^2}{10000}} \cong \pm 0.01$$

1.   Submarine on the Edge of the FOC

The submarine might decide to minimize the analytical probability of being detected by remaining on the edge of the FOC. Figure 9 shows the probability of detection for helicopter delays of 10, 20 and 30 minutes when the submarine escapes by sailing on the edge of the FOC compared with when its position is uniformly distributed in the same area.

A small but statistically significant decrease in the probability of detection is observed for helicopter's delay of 10 minutes (p value = 0.000975).
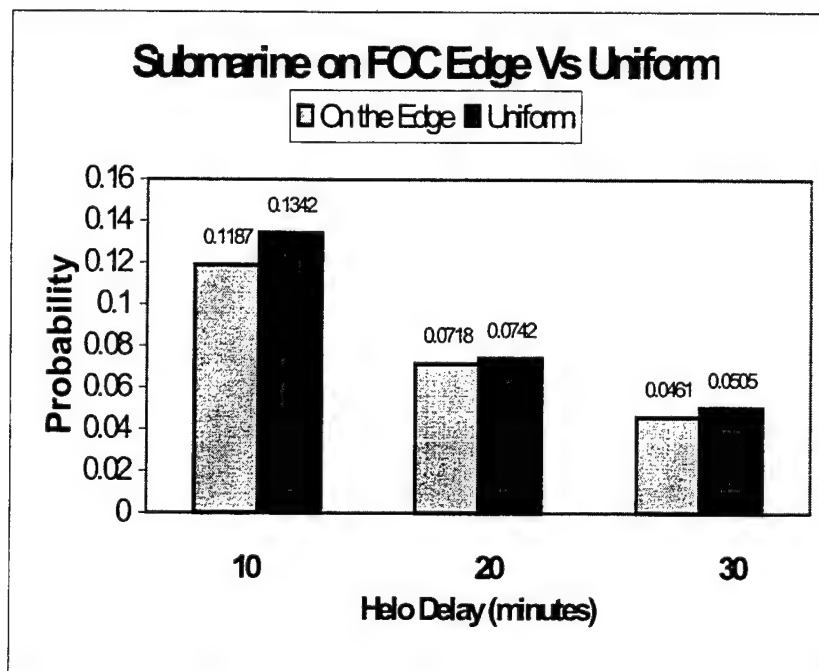
Figure 9. Submarine escaping on the edge of the FOC compared
with when its position is uniformly distributed in the same area.

This effect seems to be attractive for the submarine. However, according to our model,

this tactic has the disadvantage that at the end of the helicopter's search period the

submarine's battery is completely exhausted and its options are reduced to being

immobile, at the bottom or snorkeling. No statistically significant differences are

observed for delays of 20 and 30 minutes (p values = 0.51 and 0.15 respectively). The

tactical significance is small in all cases, which should not be surprising because the

helicopter makes no attempt to exploit the predictability of the range to the submarine.

2. Submarine Remains in the Vicinity of the Datum

The submarine might attempt to surprise the enemy force by remaining in the vicinity of the datum. This situation may be also presented if the submarine executes an involuntary attack and is forced to remain close to the datum because of its low battery level. Figure 10 shows the effectiveness of this tactic for 3 different values of the helicopter's delay compared with when the submarine behavior determines a uniformly distributed position in the FOC. If the submarine remains in the datum vicinity, no statistically significant differences are observed for delays of 10, 20 or 30 minutes (p values = 0.31, 0.43 and 0.95 respectively).
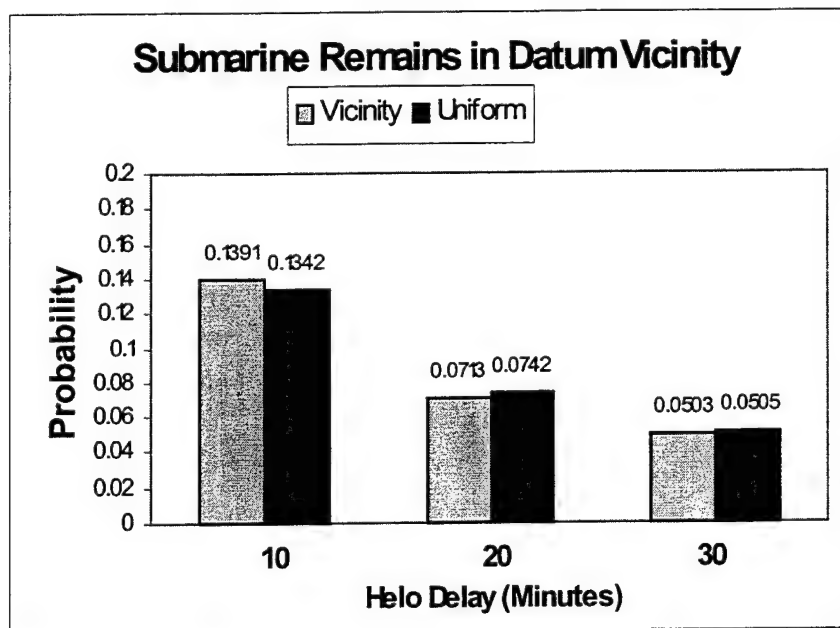
Figure 10. Probability of detection when the submarine remains
in the datum vicinity, compared with when its position is uniform.

These results indicate that the submarine does not get any tactical benefit remaining in the vicinity of the datum after the attack. Again, this should not be surprising given the helicopter's tactics.

3. Helicopter Fails First Detection Success

When the helicopter is dipping, a poorly trained crew may fail to detect the target even when the target is within range. When dealing with a diesel-electric submarine this is especially possible. They almost make no noise and most of them are small. Although it is logical to suppose that this lack of skills will negatively affect the probability of detection, our intention is to measure its real impact in the mission accomplishment. In Figure 11, a dramatic loss of probability of detection in the event that the helicopter fails to detect the first time the submarine is detectable, for all helicopter's delays.
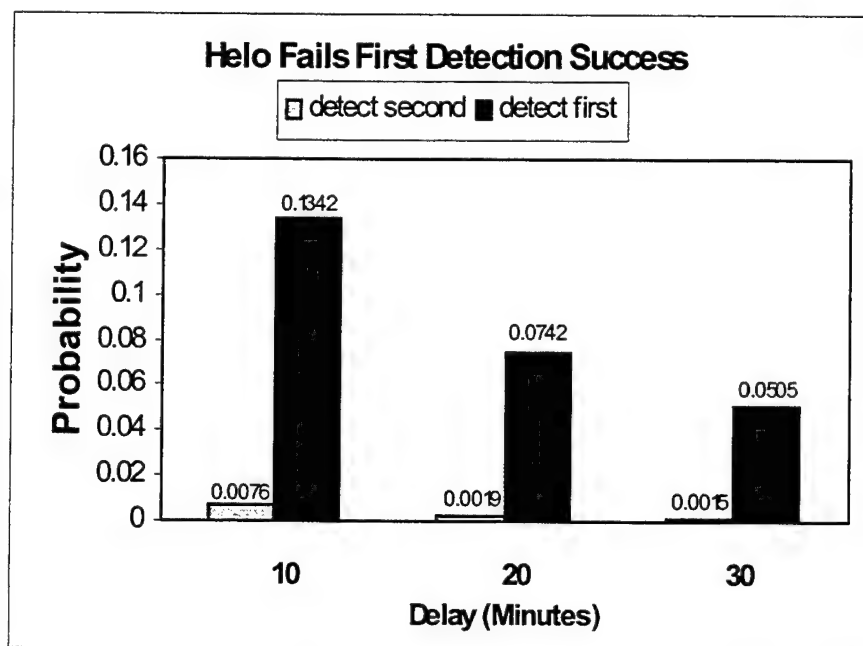
Figure 11. Probability of Detection when helicopter fails to detect the submarine when an opportunity is presented.

In this case it is very unlikely that the helicopter will ever have a second chance of detection.

4. Helicopter Searches Inside the FOC

The helicopter's search may be executed inside the circle determined by the FOC instead of the MPD. In that case, the helicopter assumes correctly that the submarine will remain inside the FOC the whole time and the search is conducted according to this assumption. When this assumption is correct, a small increase in probability of detection (p value $\cong$ 0) is observed for all three delays (Figure 12).

Searching Inside the MPD or FOC

🔲 MPD   ■ FOC
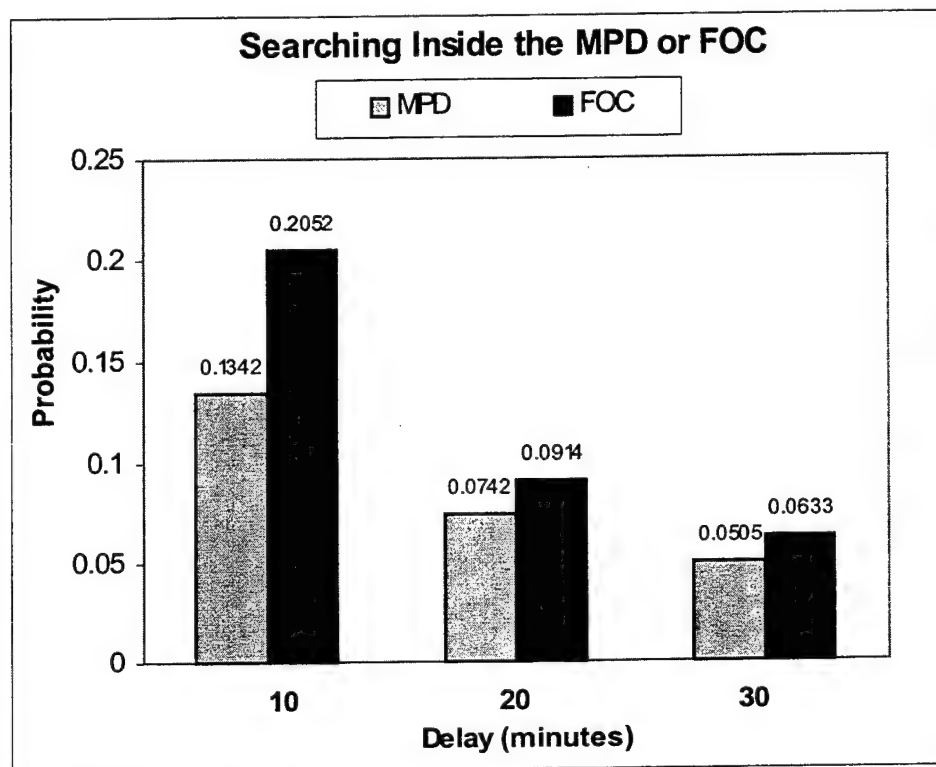
Figure 12. Probability of detection when the helicopter's search is inside the MPD or the FOC.

However, assuming that the submarine is inside the FOC will be incorrect whenever the submarine exceeds the limits of that circle. To do that the submarine could follow any of many speed behaviors determined by $u^*(t)$ given by Equation 17. For these cases

neither the analytical model nor the simulation allow us to know the effects in terms of

the probability of detection of searching based on a wrong assumption.

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V. DISCUSSIONS OF THE MODELS AND FUTURE WORK

A.   ANALYTICAL MODEL

The analytical model explored in this thesis has some evident weaknesses:

1.   Capacity of the Submarine Battery

In Chapter II it was stated that the constraint imposed by the battery's' energy if the submarine wants to move for a given period $[0, T]$ with a speed $u(t)$ is:

$$E(T) \leq E_{TOT} \qquad (Equation\ 4)$$

E_{TOT} is computed as U^3 multiplied by the endurance at this speed, where U is any arbitrarily chosen speed. For an ideal battery this should be true for every chosen U. However, batteries do not always behave in an ideal manner and therefore this method of obtaining the value of E_{TOT} is not exact. It is suggested that future studies may focus on obtaining a better model for the battery.

2.   The Edge Effect

The analytical model explained in Chapter II fails to capture the effect produced when the helicopter dips in a position where the distance to the datum S is greater than the submarine FOC minus the sonar range R. When this is the case, the detection rate decreases because the area of the sonar detection circle A(S) inside the submarine possible position disk also decreases (Figure 13). This effect is a source of inaccuracy for the model and for the probability PD_A explained as a simulation verification method in Chapter III. However, it does not seem to be very significant. Future studies may try to measure its effect in terms of probability of detection and confirm this belief.
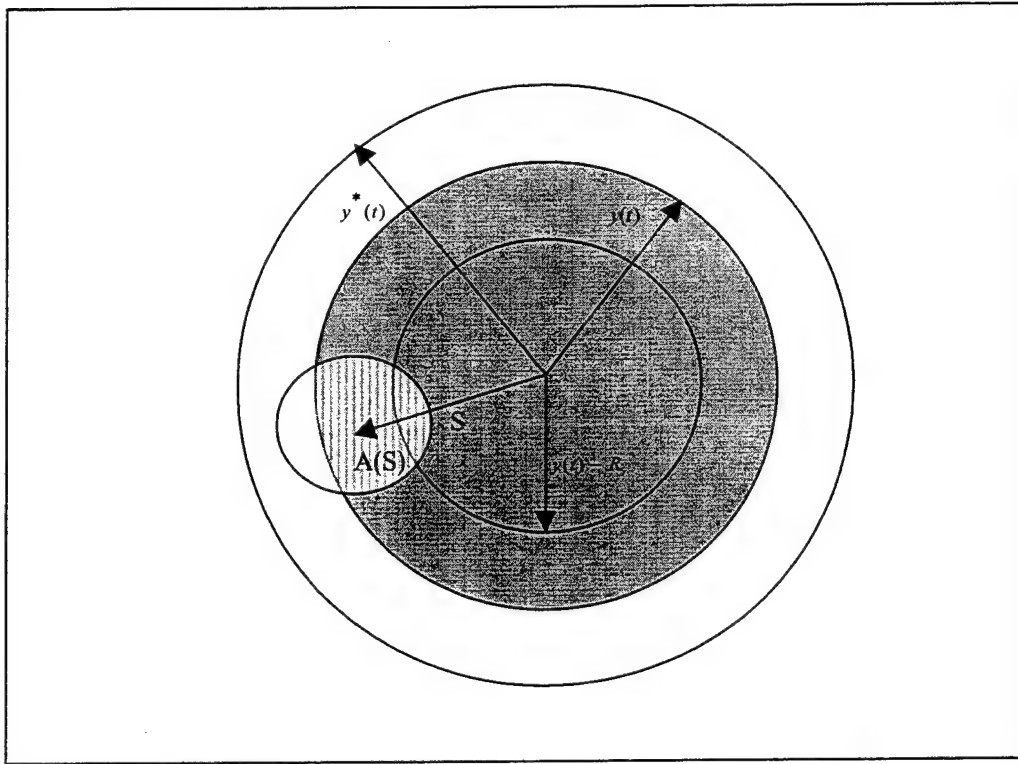
Figure 13. The Edge Effect is produced when the dip radius S is outside $y(t) - R$.

3.   The First Dip

The helicopter's first dip has a significant value in terms of probability of detection. The models studied in Chapter II gradually integrate the probability of detection during the interval $[\tau, T]$, where τ is the helicopter delay and T is the ending time of the search period. Thus, they do not capture properly that an important part of the probability is added almost immediately during the first dip. This is a source of difference between the analytic models and the simulation. Table 2 shows the probability of detection observed from 10000 replications of the simulation for different helicopter delays and the magnitude of such a probability coming from the first dip detection when

the search is conducted inside the MPD circle. It is evident that a significant percentage of the total probability of detection is because of the first dip.

Helo Delay	All dips	First dip
10	.0.1342	0.0722
15	0.0961	0.0436
20	0.0742	0.0317
25	0.0614	0.0251
30	0.505	0.0196
35	0.0437	0.0152
40	0.0355	0.0124
45	0.0274	0.0104
50	0.0264	0.0092
55	0.0225	0.0076
60	0.02	0.007
65	0.0199	0.0065
70	0.0188	0.0058
75	0.0151	0.0049

Table 2. Helicopter's First Dip.

B. FURTHER STUDIES

This study represents a more realistic method of posing the Flaming Datum problem when dealing with diesel-electric submarines that must respect an energy constraint. This constraint was never considered in any other study. For this reason the developed models may be considered a valid starting point for future works. It is suggested that they should be focused in the following aspects:

Incorporating the first dip effect as an important source of probability of detection for the analytical model.

Improve the modeling of the submarine battery. By doing so, more accurate limits for the submarine capabilities will be established and the credibility of the predictions of the two models will be increased.

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APPENDIX A. TABLE OF SIMULATION RESULTS

The table shows the probability of detection as a function of the helicopter's delay. The second column corresponds to the values obtained from the analytical model when the helicopter's dips inside the MPD area. The third, fourth and fifth columns show the values generated by the simulation and its 95 % confidence intervals. The last column is PD_A, which was used to verify the credibility of the data generated by the simulation.

Helo Delay	Analytical PD*(t)	Simulation	SIM 95% CI		PDA
10	0.107342446	0.1433	0.1533	0.1333	0.1335
15	0.075518445	0.0934	0.1034	0.0834	0.09200852
20	0.057619481	0.0726	0.0826	0.0626	0.06965769
25	0.046148474	0.0581	0.0681	0.0481	0.05553586
30	0.038182827	0.0481	0.0581	0.0381	0.04593629
35	0.032344153	0.0437	0.0537	0.0337	0.03891429
40	0.027890121	0.0355	0.0455	0.0255	0.03371097
45	0.024388764	0.0274	0.0374	0.0174	0.02958586
50	0.02157021	0.0264	0.0364	0.0164	0.0261375
55	0.019257319	0.0225	0.0325	0.0125	0.02349987
60	0.017329022	0.02	0.03	0.01	0.02116773
65	0.015700047	0.0199	0.0299	0.0099	0.01920521
70	0.014307406	0.0188	0.0288	0.0088	0.01762776
75	0.013105733	0.0151	0.0251	0.0051	0.01618498

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APPENDIX B. REGRESSION FOR THE PROBABILITY OF DETECTION

The following output shows the regression model for the probability of detection as a function of the helicopter's delay τ according to the results obtained from the simulation, when the search is executed inside the MPD.

```
        *** Linear Model ***

Call: lm(formula = probability ~ delay^3 + delay^2 + delay, dat
regression.frame, na.action
        = na.exclude)
Residuals:
     Min       1Q     Median       3Q        Max
 -0.00502 -0.001252 0.0002859 0.001655 0.005129

Coefficients:
             Value Std. Error  t value Pr(>|t|)
(Intercept)  0.2044   0.0074    27.5633  0.0000
 I(delay^3)  0.0000   0.0000    -6.4155  0.0001
 I(delay^2)  0.0002   0.0000     8.5822  0.0000
      delay -0.0090   0.0007   -13.2313  0.0000

Residual standard error: 0.003216 on 10 degrees of freedom
Multiple R-Squared: 0.9935
F-statistic: 506.2 on 3 and 10 degrees of freedom, the p-value
3.234e-011

Analysis of Variance Table

Response: probability

Terms added sequentially (first to last)
           Df   Sum of Sq      Mean Sq  F Value        Pr(F)
I(delay^3)  1 0.007579049 0.007579049 732.5684 1.096000e-010
I(delay^2)  1 0.006321668 0.006321668 611.0336 2.682000e-010
     delay  1 0.001811225 0.001811225 175.0676 1.160015e-007
 Residuals 10 0.000103459 0.000010346
```

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